L_∞-algebras

Derived deformation theory and Koszul duality

I. From dgla's to L_∞-algebras.

II. $dCoAlg_{\mathbb{K}}$: coassociative cocommutative counital coalgebras with a coderivation (ind-conilpotent)

III. Homotopy theory of L_∞-algebras.

IV. Kontsevich's theorem on deformation quantization.

References and further reading:

- Cattaneo - Indelicato, Lecture notes on Formality and Star products (arXiv: math/0403135v2)
- Hinich, DG Coalgebras as formal stacks (arXiv: math/9812034v1)
- Kontsevich, Deformation quantization of Poisson Manifolds 1 (arXiv: q-alg/9709040v1)
- Kontsevich - Soibelman, Deformation Theory 1 (available at math.ksu.edu/~soibel)
- Pridham, Unyfing derived deformation theories. (arXiv: 0705.0344v6)
- ncatlab.org (wiki-lab for collaborative work on Math!)

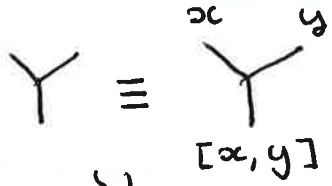
Notes of this lecture are available at Claudias Scheimbauer's site:

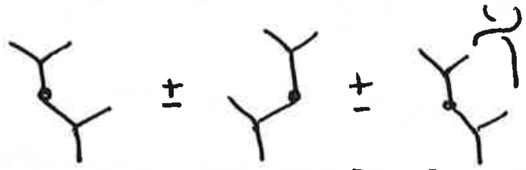
guests.mpim-bonn.mpg.de/scheimbauer/DAGX.html

Néstor León Delgado guests.mpim-bonn.mpg.de/nleon

(I). A graded Lie algebra is $(V, [-,-])$ (1)

$[-,-] : V_i \times V_j \longrightarrow V_{i+j}$ $\deg([-,-]) = 0$ (antisymmetric)



The Jacobi identity is:  $= 0$ [3]

$[[x,y],z] \pm [x,[y,z]] \pm [[x,z],y] = 0$

. In a dgla $(V, [-,-], d)$ we also have $d \equiv |$; $d^2 = 0$ ie  $= 0$ [1]

and the Leibniz rule is:  $= 0$ [2]

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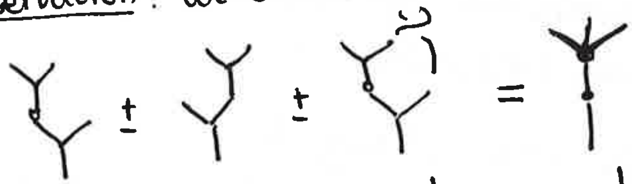
(2)

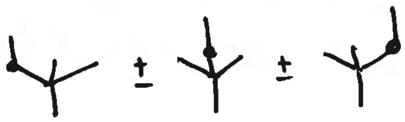
<u>Proposition</u>: If $(V, d, [-,-])$ satisfies [1], [2], [3]
Then $(H(V), [-,-])$ is a graded Lie algebra.

<u>Proof</u>: Choose x, y cocycles: $[x + dx', y] = [x,y] \pm d[x',y] \pm [x', dy = 0]$ ∎

<u>Observation</u>: we could also have Jacobi up to an exact term = 'up to homotopy'

 [3']

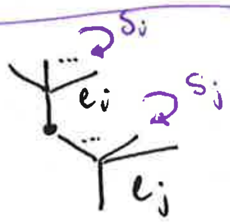
Where $l_3 \equiv$  $: V \times V \times V \longrightarrow V$
$|l_3| = -1$ antisym.

And even  $+$ RHS[3'] $=$  [3'']

It is clear that if we upgrade the Proposition changing [3] by [3'] or [3''] the result still holds

(3)

We can see a pattern

$l_i : V\times \overset{i}{\cdots} \times V \longrightarrow V$; $|l_i| = 2-i$ $\qquad \sum_{i+j=n+1} \pm$  $= 0$

<u>Definition</u>: An L_∞-algebra $(V, \{l_i\}_{i\geq 1})$ is a graded vec. space V and a family of maps $l_i : V\otimes \overset{i}{\cdots} \otimes V \longrightarrow V$ antisymmetric $|l_i| = 2-i$ such that $\forall\, v_1\otimes \dots \otimes v_n \in V^{\otimes n}$ and $\forall n\in\mathbb{N}$

$$0 = \sum_{i+j=n+1} (-1)^{j(i-1)} \sum_{\sigma\in Sh_i^j} \epsilon(\sigma,v)(-1)^{\sigma}\, l_j(l_i(v_{\sigma(1)},\dots,v_{\sigma(i)}), v_{\sigma(i+1)},\dots, v_{\sigma(n)}) \; [n]$$

$\longrightarrow$ Koszul sign if $\sigma = (1,2)$ $\epsilon(\sigma, v) = (-1)^{|v_1||v_2|}$.

<u>Examples</u>: cochain complexes, graded lie algebras, dgla's.

Once again, we can upgrade the Proposition, if $(V, \{l_i\})$ is an L_∞-algebra then $(H(V), l_2)$ is a graded Lie algebra.

(4)

Examples: central extensions $(h \longrightarrow g' \longrightarrow g$ s.e.s.$)$

- Let $(g, [-,-])$ be a Lie algebra, $\mu \in (\Lambda^2 g)^*$ a 2-cocycle $g_\mu := (g\oplus \mathbb{R}, [-,-]\oplus\mu)$ is a 1-dimensional central extension

  $\mathbb{R} \longrightarrow g_\mu \longrightarrow g$

- In general, if we consider $\mu\in(\Lambda^{i+1} g)^*$ we get a 1-dimensional central extension of L_∞-algebras $\mathbb{R}[i-1] \longrightarrow g_\mu \longrightarrow g$

  $g_\mu = (g\oplus \mathbb{R}[i-1],\ l_2 = [-,-],\ l_{i+1} = \mu)$

- If g is semisimple, $\langle -,-\rangle$ its killing form, $\mu := \langle -, [-,-]\rangle \in (\Lambda^3 g)^*$ is called a string extension.

We still don't know about morphisms of L_∞-algebras (they are not the naïve ones $l_i(f,\dots,f) = f\, l_i(-,\dots,-)$.

We will learn about them after we know something about coalgebras.

II **Definition**: $CoAlg_K$ denotes the category with objects (X, Δ) $X \in grVec$, $\Delta : X \longrightarrow X \otimes X$ and $f : X \longrightarrow Y$ morphisms s.t. ⑤

$$\begin{tikzcd} X\otimes X\otimes X & X\otimes X \arrow[l] \\ X\otimes X \arrow[u] & X \arrow[l] \arrow[u] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \qquad \begin{tikzcd} X\otimes X \arrow[rr, no head] & & X\otimes X \\ & X \arrow[ul] \arrow[ur] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] \\ \mathbb{K} \end{tikzcd} \; ; \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ X\otimes X \arrow[r] & Y\otimes Y \end{tikzcd}$$

co associative — co commutative — counital — morphisms

Technical remark: we want only ind-conilpotent coalgebras i.e. $F^n := \ker(\Delta^n : X \longrightarrow X^{\otimes n})$ is an exhaustive filtration.

· There are some diagrams missing of the compatibility of the counit, but they are dual to the ones in unital algebras.

· Some authors call ind-conilpotent, connected

Example: cofree coalgebra, there is an adjunction ⑥

Forget : $CoAlg \rightleftarrows gVec : C$ where

$$C(V) = \bigoplus_{n \geq 0} \left(S^n(V) = (V^{\otimes n})^{S_n} \right)$$ [the invariants]

$$\Delta(v_1 \otimes .. \otimes v_n) = \sum_{\sigma \in S_n} \sum_{i=0}^{n} \frac{1}{n!} (v_{\sigma(1)} \otimes .. \otimes v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \otimes .. \otimes v_{\sigma(n)})$$

Definition: $dCoAlg_K$ is given by (X, Q) where $X \in CoAlg$ $Q : X_i \longrightarrow X_{i+1}$ $Q^2 = 0$ and Q is a coderivation i.e. $\Delta \circ Q = (Q \otimes id \pm id \otimes Q) \circ \Delta$ with morphisms s.t. $f \circ Q = Q \circ f$.

Proposition: If $(V, \{l_i\})$ is an L_∞-algebra

Then $(C(V[1]), \oplus l_i)$ is a $dCoAlg_{\mathbb{K}}$

Idea of the proof: $Q^2 = 0 \iff [n] \; \forall n \in \mathbb{N}$.

And the contrary, any Q in $C(V[1])$ corresponds to $\{l_i\}$ in V.

Definition: A morphism of L_∞-algebras is a morphism of the associated $dCoAlg_{\mathbb{K}}$. i.e.

$$\begin{aligned} L_\infty\text{-}alg_{\mathbb{K}} &\longrightarrow dCoAlg_{\mathbb{K}} \\ V &\longmapsto C(V[1]) \end{aligned}$$

is fully-faithfull

Observation: the functor $dgla_{\mathbb{K}} \longrightarrow L_\infty\text{-}alg_{\mathbb{K}}$ is not full

i.e. there are more L_∞-morphisms between dgla's than dgla morphisms

Only if there is time before the break:

Extra: geometric interpretation of cofree coalgebras: formal manifolds

Given $x \in M$ a point in a smooth manifold, we can consider smooth functions around x (i.e. smooth functions around 0 in $V \equiv \mathbb{R}^m$)

Taylor expansion $\quad f(x) = \sum_{n \geq 0} \sum_{I, |I| = n} \frac{x^I}{I!} \partial^I f(0)$

$$f \in \prod_{n \geq} Hom(S^n V, \mathbb{R}) \cong Hom(\bigoplus_{n \geq 0} S^n V, \mathbb{R}) = (C(V))^\vee$$

Think of the coalgebra of distributions (dual to functions)

This motivates the choice of coderivations (dual to vector fields.

→ BREAK ←

III

Recall: Chevalley-Eilenberg homology functor

$$\begin{array}{ccc} dCoAlg_k & \xleftarrow{CE_*} & dgla_k \\ (C(V[1]), Q) & \longleftarrow\!\shortmid & (V, d, [-,-]) \end{array}$$

where Q extends $d + [-,-]$ as a coderivation.

There is a left-adjoint
$$\begin{array}{ccc} dCoAlg_k & \xrightarrow{\ell} & dgla_k \\ (X, \Delta, Q) & \longmapsto & (F(X[-1]), d) \end{array}$$
d extends $(Q+\Delta)|_{X[-1]}$ by derivation

Theorem (Hinich): There is a model structure on $dCoAlg_k$ such that $\ell \dashv CE_*$ is a Quillen equivalence.

$WE = \{ f : \ell(f) \text{ is a w.e.} \}$ $\quad COFIB = \{ f : \text{degreewise injective} \}$.

$$\begin{tikzcd} & & dCoAlg_k \arrow[d, shift left, "*"] \\ dgla_k \arrow[urr, shift left, "CE_*", "\cong"'] \arrow[r] \arrow[drr, "CE^*"'] & L_\infty\text{-alg}_k \arrow[ur] \arrow[l, shift left] \arrow[dr] & f\text{-}Alg_k \arrow[u, shift left, "*"] \end{tikzcd}$$

There is no model structure for L_∞-algebras but:
$$L_\infty\text{-alg}_k[W^{-1}] \underset{\text{as }(\infty,1)\text{-cat.}}{\simeq} dgla_k[W^{-1}]$$

1. CE^* has values in algebras, but there is a category $f\text{-}Alg_k$ s.t. $A^* \in dCoAlg$ $\forall A \in f\text{-}Alg_k$ which is more interesting
2. Theorem: taking duals is a Quillen equivalence.
3. Both CE_* and CE^* factor through L_∞-algebras.
4. L_∞-algebras are the fibrant objects in $f\text{-}Alg_k$
5. Taking Hammock localization $L_\infty\text{-alg}_k[W^{-1}] \simeq f\text{-}Alg_k[W^{-1}] \simeq dgla_k[W^{-1}]$

ℓ has an interesting geometric interpretation, thinking of (X, Q) as a formal manifold (with codifferential) $\ell(X, Q)$ is the tangent Lie algebra (only that also equipped with a differential)

IV

10

Classical Mechanics	Quantum Mechanics
(M,π) Poisson manifold (phase space: position-momentum)	$\mathcal{H}$. $\mathbb{C}$-Hilbert space (fields) ~~replace~~
$H \in C^\infty(M)$ hamiltonian function	$\hat{H}$ operator on $\mathcal{H}$
Physical states: points in M	Physical states: vectors in $\mathcal{H}$
Observables: $C^\infty(M)$ (energy ...)	Observable: self adjoint operators
Time evolution $\frac{df}{dt} = \{H, f\}$ $\{-,-\}$ Poisson bracket	Time evolution: $\frac{d\hat{f}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{f}]$ $[-,-]$ commutator.

$\hbar \to 0$ geometric / deformation quant.

11

Geometric quantization: construct $\mathcal{H}$ from $C^\infty(\mathbb{R}^{2n})$

Deformation quantization: deform the commutative product in $C^\infty(\mathbb{R}^{2n})$ to a non-commutative in $\mathcal{H}$: STAR PRODUCTS.

Idea: consider deformations of both structures

they are given by dgla's $\mathfrak{g}$.

Construct the associated def. functors $\mathrm{Artin}_k \longrightarrow \mathrm{Sets}$

$A \longmapsto MC(A \otimes \mathfrak{g})$

Compare the two sides.

12

Definition: A Poisson manifold is (M,π) a smooth manifold M and $\pi \in \Gamma(M, \Lambda^2 TM)$ such that $[\pi,\pi]_{SN} = 0$ $\Leftrightarrow$ Jacobi

• π induces a bracket on $C^\infty(M)$ via $\{f,g\} := \frac{1}{2}\langle \pi, df \wedge dg\rangle$

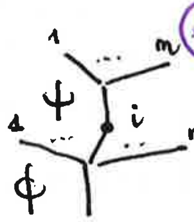
Definition: Schouten-Nijenhuis dgla of multivector fields is $(V^i := \Gamma(M, \Lambda^{i+1} TM),\ d=0,\ [-,-]_{SN})$ where

$$[x_1\wedge\ldots\wedge x_n, y_1\wedge\ldots\wedge y_m]_{SN} = \sum (-1)^{i+j} [x_i, y_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n \wedge y_1 \wedge \ldots \wedge \hat{y}_j \ldots y_m$$

• $\pi \in MC(V) \Leftrightarrow \pi \in \Gamma(M, \Lambda^2 TM)$ and $[\pi,\pi] = 0 \Leftrightarrow \pi$ is a Poisson structure

Definition: Hochschield dgla. Let A be an associative unital $\mathbb{K}$-algebra. 13

$\left(C^i := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A),\ [-,-]_G,\ [m,-]\right)$

$[\phi,\psi]_G := \phi \bullet \psi \pm \psi \bullet \phi$, $\phi \bullet_i \psi = (-1)^i \phi \circ_i \psi$; $\phi \circ_i \psi =$ 

In our case $A = C^\infty(M)$, $D^i \subset C^i$ consists of multidifferential operators $\left(\phi(f,g) = \sum_{I,J} \phi^{IJ} \partial_I f \partial_J g\right.$ and so on$\left.\right)$ such that $\phi(1,\dots) = 0$ (technical conditions for star products)

Observe m associative $\Leftrightarrow [m,m] = 0$;

$\phi \in MC \Leftrightarrow [m,\phi] + \frac{1}{2}[\phi,\phi] = 0 \Leftrightarrow [m+\phi, m+\phi] = 0$

Theorem HKR (Hochschield-Konstant-Rosenberg) 14

$$V = H(V) \xleftarrow{\cong} H(D)$$

Theorem (Kontsevich) $V \xleftarrow{\simeq} D$ quasi-isomorphism of L_∞-algebras

(locally, the proof is for $\mathbb{R}^d$)

This means that the deformation problems are equivalent.

Even more, taking formal deformations we get a similar result

$V[[\hbar]] \xleftarrow{\simeq} D[[\hbar]]$ in an inductive way

↳ Formal deformation of Poisson manifolds

↳ Star products: deformations of m.